10/8/2020

B.Sc. Part I (Hons). 1st Paper

Trigonometry (Summation of Series) (contd)

1. Find the sum of the following series

$\cos\alpha \sin\alpha + \frac{1}{2}\cos^2\alpha \sin 2\alpha + \frac{1}{3}\cos^3\alpha \sin 3\alpha + \cdots$ to ∞

Soln: Let the given series be denoted by S.

$\Rightarrow S = \cos\alpha \sin\alpha + \frac{1}{2}\cos^2\alpha \sin 2\alpha + \frac{1}{3}\cos^3\alpha \sin 3\alpha + \cdots$ to ∞

Let

$C = \cos\alpha \cos\alpha + \frac{1}{2}\cos^2\alpha \cos 2\alpha + \frac{1}{3}\cos^3\alpha \cos 3\alpha + \cdots$ to ∞

$\Rightarrow C + iS = \cos\alpha(\cos\alpha + i\sin\alpha) + \frac{1}{2}\cos^2\alpha(\cos 2\alpha + i\sin 2\alpha)$
$+ \frac{1}{3}\cos^3\alpha(\cos 3\alpha + i\sin 3\alpha) + \cdots$ to ∞

$\Rightarrow C + iS = \cos\alpha \cdot e^{i\alpha} + \frac{1}{2}\cos^2\alpha \cdot e^{2i\alpha} + \frac{1}{3}\cos^3\alpha\, e^{3i\alpha}$
$+ \cdots$ to ∞

Put $\cos\alpha \cdot e^{i\alpha} = x$

$\Rightarrow C + iS = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ to ∞

$\Rightarrow C + iS = -\log(1-x) = -\log\{1 - \cos\alpha \cdot e^{i\alpha}\}$

$\Rightarrow C + iS = -\log\{1 - \cos\alpha(\cos\alpha + i\sin\alpha)\}$

$$\Rightarrow C+iS = -\log\{1-\cos^2\alpha + (-)\, i\sin\alpha\cos\alpha\}$$

$$= -\log\{\sin^2\alpha - i\sin\alpha\cos\alpha\}$$

$$\cancel{= -\log\{\sin\alpha(\sin\alpha - i\cos\alpha)\}}$$

$$\cancel{= -\log\left[\sin\alpha.\{-i^2\sin\alpha - i\cos\alpha\}\right]}$$

$$\cancel{= -\log\left[-i\sin\alpha.(\cos\alpha + i\sin\alpha)\right]}$$

$$\Rightarrow C+iS = -\frac{1}{2}\log(\sin^4\alpha + \sin^2\alpha\cos^2\alpha) + i\tan^{-1}\frac{\sin\alpha\cos\alpha}{\sin^2\alpha}$$

$$\Rightarrow C+iS = -\frac{1}{2}\log\left[\sin^2\alpha(\sin^2\alpha+\cos^2\alpha)\right] + i\tan^{-1}(\cot\alpha)$$

$$= -\frac{1}{2}\log\left[\sin^2\alpha\right] + i\tan^{-1}\{\tan(\tfrac{\pi}{2}-$$

$$\Rightarrow C+iS = -\log\sin\alpha + i\left(\frac{\pi}{2}-\alpha\right)$$

Equating imaginary parts, we get

$$\Rightarrow S = \frac{\pi}{2} - \alpha.$$